

A cellular algebra with specific decomposition of the unity

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Abstract. Let \mathbb{A} be a cellular algebra over a field \mathbb{F} with a decomposition of the identity $1_{\mathbb{A}}$ into orthogonal idempotents e_i , $i \in I$ (for some finite set I) satisfying some properties. We describe the entire Loewy structure of cell modules of the algebra \mathbb{A} by using the representation theory of the algebra $e_i \mathbb{A} e_i$ for each i . Moreover, we also study the block theory of \mathbb{A} by using this decomposition.

1 Introduction

Let $\mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1})$, or simply $\mathbb{T}_{n,m}$, be the bubble algebra with m -different colours, $\delta_i \in \mathbb{F}$, which is defined in Grimm and Martin[2]. In the same paper, it has been showed that it is semi-simple when none of the parameters δ_i is a root of unity. Later, Jegan[4] showed that the bubble algebra is a cellular algebra in the sense of Graham and Lehrer[1]. The identity of the algebra $\mathbb{T}_{n,m}$ is the summation of all the different multi-colour partitions that their diagrams connect j only to j' with any colour for each $1 \leq j \leq n$, these multi-colour partitions are orthogonal idempotents. The goal of this paper is to generalize the technique that we use to study the representation theory of $\mathbb{T}_{n,m}$ in [3].

Wada[8] consider a decomposition of the unit element into orthogonal idempotents and a certain map α to define a Levi type subalgebra and a parabolic subalgebra of the algebra \mathbb{A} , and then the relation between the representation theory of each one has been studied. With using the same decomposition, we construct a Levi type subalgebra $\bar{\mathbb{A}}$ (without using any map), and classify the blocks of \mathbb{A} by using the representation theory of the algebra $\bar{\mathbb{A}}$.

2 Cellular algebras

We start by reviewing the definition of a *cellular algebra*, which was introduced by Graham and Lehrer[1] over a ring but we replace it by a field, since we need this assumption later.

Definition 2.1. [1, Definition 1.1]. A cellular algebra over \mathbb{F} is an associative unital algebra \mathbb{A} , together with a tuple $(\Lambda, T(\cdot), \mathbb{C}, *)$ such that

1. The set Λ is finite and partially ordered by the relation \geq .
2. For every $\lambda \in \Lambda$, there is a non-empty finite set $T(\lambda)$ such that for an pair $(s, t) \in T(\lambda) \times T(\lambda)$ we have an element $c_{st}^\lambda \in \mathbb{A}$, and the set $\mathbb{C} := \{c_{st}^\lambda \mid s, t \in T(\lambda) \text{ for some } \lambda \in \Lambda\}$ forms a free \mathbb{F} -basis of \mathbb{A} .

3. The map $*$: $\mathbb{A} \rightarrow \mathbb{A}$ is \mathbb{F} -linear involution (This means that $*$ is an anti-automorphism with $*^2 = id_{\mathbb{A}}$ and $(c_{st}^\lambda)^* = c_{ts}^\lambda$ for all $\lambda \in \Lambda$, $s, t \in T(\lambda)$).
4. For $\lambda \in \Lambda$, $s, t \in T(\lambda)$ and $a \in \mathbb{A}$ we have

$$ac_{st}^\lambda \equiv \sum_{u \in T(\lambda)} r_a^{(u,s)} c_{ut}^\lambda \pmod{\mathbb{A}^{>\lambda}}, \quad (1)$$

where $r_a^{(u,s)} \in \mathbb{F}$ depends only on a, u and s . Here $\mathbb{A}^{>\lambda}$ denotes the \mathbb{F} -span of all basis elements with upper index strictly greater than λ .

For each $\lambda \in \Lambda$, the cell module $\Delta(\lambda)$ is the left \mathbb{A} -module with an \mathbb{F} -basis $\mathbf{B} := \{c_s^\lambda \mid s \in T(\lambda)\}$ and an action defined by

$$ac_s^\lambda = \sum_{u \in T(\lambda)} r_a^{(s,u)} c_u^\lambda \quad (a \in \mathbb{A}, s \in T(\lambda)), \quad (2)$$

where $r_a^{(s,u)} \in \mathbb{F}$ is the same coefficient that in (1).

A bilinear form $\langle \cdot, \cdot \rangle : \Delta(\lambda) \times \Delta(\lambda) \rightarrow \mathbb{F}$ can be defined by

$$\langle c_s^\lambda, c_t^\lambda \rangle c_{ub}^\lambda \equiv c_{us}^\lambda c_{tb}^\lambda \pmod{\mathbb{A}^{>\lambda}} \quad (s, t, u, b \in T(\lambda)).$$

Note that this definition does not depend on the choice of $u, b \in T(\lambda)$.

Let \mathbf{G}_λ be the Gram matrix for $\Delta(\lambda)$ of the previous bilinear form with respect to the basis \mathbf{B} . All Gram matrices of cell modules that will be mentioned in this paper are with respect to the basis \mathbf{B} with the bilinear form defined by (2).

Let Λ^0 be the subset $\{\lambda \in \Lambda \mid \langle \cdot, \cdot \rangle \neq 0\}$. The radical

$$\text{Rad}(\Delta(\lambda)) = \{x \in \Delta(\lambda) \mid \langle x, y \rangle = 0 \text{ for any } y \in \Delta(\lambda)\}$$

of the form $\langle \cdot, \cdot \rangle$ is an \mathbb{A} -submodule of $\Delta(\lambda)$.

Theorem 2.2. [6, Chapter 2]. Let \mathbb{A} be a cellular algebra over a field \mathbb{F} . Then

1. \mathbb{A} is semi-simple if and only if $\det \mathbf{G}_\lambda \neq 0$ for each $\lambda \in \Lambda$.
2. The quotient module $\Delta(\lambda)/\text{Rad}(\Delta(\lambda))$ is either irreducible or zero. That means that $\text{Rad}(\Delta(\lambda))$ is the radical of the module $\Delta(\lambda)$ if $\langle \cdot, \cdot \rangle \neq 0$.
3. The set $\{L(\lambda) := \Delta(\lambda)/\text{Rad}(\Delta(\lambda)) \mid \lambda \in \Lambda^0\}$ consists of all non-isomorphic irreducible \mathbb{A} -modules.
4. Each cell module $\Delta(\lambda)$ of \mathbb{A} has a composition series with sub-quotients isomorphic to $L(\mu)$, where $\mu \in \Lambda^0$. The multiplicity of $L(\mu)$ is the same in any composition series of $\Delta(\lambda)$ and we write $d_{\lambda\mu} = [\Delta(\lambda) : L(\mu)]$ for this multiplicity.
5. The decomposition matrix $\mathbf{D} = (d_{\lambda\mu})_{\lambda \in \Lambda, \mu \in \Lambda^0}$ is upper uni-triangular, i.e. $d_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ and $d_{\lambda\lambda} = 1$ for $\lambda \in \Lambda^0$.
6. If Λ is a finite set and \mathcal{C} is the Cartan matrix of \mathbb{A} , then $\mathcal{C} = \mathbf{D}^t \mathbf{D}$.

3 A Levi type sub-algebra

In this section, we construct a Live type subalgebra $\bar{\mathbb{A}}$ of \mathbb{A} and study its representation theory.

The second and the third parts of the following assumption existed in Assumption 2.1 in [8].

Assumption 3.1. *Throughout the remainder of this paper, we assume the following statements (A1) – (A4).*

(A1) *There exists a finite set I .*

(A2) *The unit element $1_{\mathbb{A}}$ of \mathbb{A} is decomposed as $1_{\mathbb{A}} = \sum_{i \in I} e_i$ with $e_i \neq 0$ and $e_i e_j = 0$ for all $i \neq j$ and $e_i^2 = e_i$.*

(A3) *For each $\lambda \in \Lambda$ and each $t \in T(\lambda)$, there exists an element $i \in I$ such that*

$$e_i c_{ts}^\lambda = c_{ts}^\lambda \quad \text{for any } s \in T(\lambda). \quad (3)$$

(A4) *$e_i^* = e_i$ for each $i \in I$. Note that from (3), for each $\lambda \in \Lambda$ and each $t \in T(\lambda)$ we have*

$$c_{st}^\lambda e_i = c_{st}^\lambda e_i^* = (e_i c_{ts}^\lambda)^* = (c_{ts}^\lambda)^* = c_{st}^\lambda \quad \text{for any } s \in T(\lambda). \quad (4)$$

From (A2) and (A3), we obtain the next lemma.

Lemma 3.2. [8, Lemma 2.2]. *Let $t \in T(\lambda)$, where $\lambda \in \Lambda$, and $i \in I$ be such that $e_i c_{ts}^\lambda = c_{ts}^\lambda$ for any $s \in T(\lambda)$. Then for any $j \in I$ such that $j \neq i$, we have $e_j c_{ts}^\lambda = 0$ for any $s \in T(\lambda)$. In particular, for each $t \in T(\lambda)$, there exists a unique $i \in I$ such that $e_i c_{ts}^\lambda = c_{ts}^\lambda$ for any $s \in T(\lambda)$.*

For $\lambda \in \Lambda$ and $i \in I$, we define

$$\begin{aligned} T(\lambda, i) &= \{t \in T(\lambda) \mid e_i c_{ts}^\lambda = c_{ts}^\lambda \text{ for any } s \in T(\lambda)\}, \\ \Lambda_i &= \{\lambda \in \Lambda \mid T(\lambda, i) \neq \emptyset\}, \\ I_\lambda &= \{i \in I \mid \lambda \in \Lambda_i\}. \end{aligned}$$

By Lemma 3.2, we have

$$T(\lambda) = \coprod_{i \in I} T(\lambda, i).$$

Note that Λ_i is a poset with the same order relation on Λ and $\Lambda = \bigcup_{i \in I} \Lambda_i$. Moreover, $\Lambda_i \neq \emptyset$ for each $i \in I$, and that because of $0 \neq e_i \in \mathbb{A}$ and $e_i^2 = e_i$.

From (A3) and Lemma 3.2, the element e_i can be written in the form

$$\sum_{\substack{\lambda \in \Lambda_i \\ s, t \in T(\lambda, i)}} b_{(s, t, \lambda)} c_{st}^\lambda$$

where $b_{(s, t, \lambda)} \in \mathbb{F}$.

Theorem 3.3. *The algebra $e_i\mathbb{A}e_i$ is a cellular algebra with a cellular basis $C_i := \{c_{st}^\lambda \mid s, t \in T(\lambda, i) \text{ for some } \lambda \in \Lambda_i\}$ with respect to the poset Λ_i and the index set $T(\lambda, i)$ for $\lambda \in \Lambda_i$, i.e. the following property holds;*

(1) *An \mathbb{F} -linear map $*$: $e_i\mathbb{A}e_i \rightarrow e_i\mathbb{A}e_i$ defined by $c_{st}^\lambda \mapsto c_{ts}^\lambda$ for all $c_{st}^\lambda \in C_i$ gives an algebra anti-automorphism of $e_i\mathbb{A}e_i$.*

(2) *For any $a \in e_i\mathbb{A}e_i$, $c_{st}^\lambda \in C_i$, we have*

$$ac_{st}^\lambda \equiv \sum_{u \in T(\lambda, i)} r_a^{(u, s)} c_{ut}^\lambda \pmod{(e_i\mathbb{A}e_i)^{>\lambda}},$$

where $(e_i\mathbb{A}e_i)^{>\lambda}$ is an \mathbb{F} -submodule of $e_i\mathbb{A}e_i$ spanned by $\{C_{st}^{\lambda'} \mid s, t \in T(\lambda', i) \text{ for some } \lambda' \in \Lambda_i \text{ such that } \lambda' > \lambda\}$, and $r_a^{(u, s)}$ does not depend on the choice of $t \in T(\lambda, i)$.

Proof. Since C is a basis of \mathbb{A} , $e_i a e_i = a$ for all $a \in e_i\mathbb{A}e_i$ and

$$e_i c_{st}^\lambda e_i = \begin{cases} c_{st}^\lambda & \text{if } s, t \in T(\lambda, i), \\ 0 & \text{otherwise,} \end{cases}$$

so the set C_i is a basis of the algebra $e_i\mathbb{A}e_i$. The first part follows from the fact the map $*$ on the algebra \mathbb{A} leaves $e_i\mathbb{A}e_i$ invariant. For the second part, from (1) we have

$$ac_{st}^\lambda \equiv \sum_{u \in T(\lambda)} r_a^{(u, s)} c_{ut}^\lambda \pmod{\mathbb{A}^{>\lambda}},$$

where $r_a^{(u, s)} \in \mathbb{F}$ depends only on a, u and s . But $e_i a = a$, so

$$ac_{st}^\lambda \equiv \sum_{u \in T(\lambda)} r_a^{(u, s)} e_i c_{ut}^\lambda \pmod{\mathbb{A}^{>\lambda}} = \sum_{u \in T(\lambda, i)} r_a^{(u, s)} c_{ut}^\lambda \pmod{\mathbb{A}^{>\lambda}}.$$

Also by using Lemma 3.2 we can show $e_i \mathbb{A}^{>\lambda} e_i = (e_i\mathbb{A}e_i)^{>\lambda}$, we are done. Moreover, cell modules $V(\lambda, i)$ for the algebra $e_i\mathbb{A}e_i$ can be defined as follows:

$$V(\lambda, i) := e_i \Delta(\lambda) \quad (\lambda \in \Lambda_i).$$

The set $B_i := \{c_s^\lambda \mid s \in T(\lambda, i)\}$ is a basis of the module $V(\lambda, i)$. □

Define the algebra $\bar{\mathbb{A}}$ to be $\sum_{i \in I} e_i\mathbb{A}e_i$ (which is the same as $\bigoplus_{i \in I} e_i\mathbb{A}e_i$ since $e_i e_j = 0$ for all $i \neq j$). The identity of the algebra $e_i\mathbb{A}e_i$ is the idempotent e_i , so $\bar{\mathbb{A}} \hookrightarrow \mathbb{A}$. Moreover, the algebra $\bar{\mathbb{A}}$ turns out to be cellular with cell modules:

$$V(\lambda, i) = e_i \Delta(\lambda) \quad (\lambda \in \Lambda_i, i \in I).$$

We put $V(\lambda, i) = \{0\}$ in the case λ is not an element in Λ_i .

Lemma 3.4. *Let $\lambda \in \Lambda$, then*

$$\Delta(\lambda) = \bigoplus_{i \in I} V(\lambda, i)$$

as an $\bar{\mathbb{A}}$ -module.

Proof. It comes directly from the fact that $1_{\mathbb{A}} = \sum_{i \in I} e_i$ and $e_i e_j = 0$ if $i \neq j$. □

4 Idempotent localization

In this section we compute the radical and Gram matrix of each cell module of the algebra \mathbb{A} by using the ones of the algebra $\bar{\mathbb{A}}$.

Let $c_{us}^\lambda, c_{tv}^\lambda \in \mathbb{C}$ where $s \in T(\lambda, i)$ and $t \in T(\lambda, j)$ for some $i, j \in I$. If $i \neq j$, then $c_{us}^\lambda c_{tv}^\lambda = 0$ which means $\langle c_s^\lambda, c_t^\lambda \rangle = 0$ in $\Delta(\lambda)$. If $i = j$, then

$$c_{us}^\lambda c_{tv}^\lambda \equiv \langle c_s^\lambda, c_t^\lambda \rangle c_{uv}^\lambda \pmod{\mathbb{A}^{>\lambda}}.$$

Since u, v do not have a role here, we can assume $u, v \in T(\lambda, i)$ and then

$$c_{us}^\lambda c_{tv}^\lambda \equiv \langle c_s^\lambda, c_t^\lambda \rangle c_{uv}^\lambda \pmod{(e_i \mathbb{A} e_i)^{>\lambda}}.$$

Hence the inner product $\langle c_s^\lambda, c_t^\lambda \rangle$ in $\Delta(\lambda)$ and the inner product $\langle c_s^\lambda, c_t^\lambda \rangle$ in $V(\lambda, i)$ have the same value. Let $M(\lambda, i)$ be the Gram matrix of this inner product on the module $V(\lambda, i)$ with respect to the basis B_i , then

$$G(\lambda) = \bigoplus_{i \in I_\lambda} M(\lambda, i). \quad (5)$$

We can show the previous result by using the facts $B = \coprod_{i \in I} B_i$, $B_i \cap B_j = \emptyset$ whenever $i \neq j$ and $\langle x, y \rangle = 0$ in $\Delta(\lambda)$ whenever $x \in V(\lambda, i)$, $y \in V(\lambda, j)$ where $i \neq j$.

The previous equation show that $\det G(\lambda) \neq 0$ if and only if $\det M(\lambda, i) \neq 0$ for each $i \in I$ such that $\lambda \in \Lambda_i$, then the following fact is straightforward.

Theorem 4.1. *The algebra \mathbb{A} is semi-simple if and only if the algebra $e_i \mathbb{A} e_i$ is semi-simple for each i .*

Proof. It comes directly from (5) and from Theorem 2.2. \square

Lemma 4.2. *Let $\lambda \in \Lambda^0$. The head of the module $\Delta(\lambda)$, denoted by $L(\lambda)$, satisfies the relation*

$$\dim L(\lambda) = \sum_{i \in I_\lambda} \dim \bar{V}(\lambda, i),$$

where $\bar{V}(\lambda, i)$ is the head of the $e_i \mathbb{A} e_i$ -module $V(\lambda, i)$. We put $\dim \bar{V}(\lambda, i) = 0$ if λ is not contained in Λ_i^0 .

Proof. This follows from the fact that $\dim L(\lambda) = \text{rank}(G(\lambda))$ as the algebra is over a field and $\lambda \in \Lambda^0$ and by using (5). \square

Theorem 4.3. *Let $\lambda \in \Lambda$, then*

$$\text{Rad}(\Delta(\lambda)) \cong \bigoplus_{i \in I_\lambda} \text{Rad}(V(\lambda, i))$$

as a vector space and

$$\text{Rad}(\Delta(\lambda)) \cong \sum_{i \in I_\lambda} \text{Rad}(V(\lambda, i))$$

as an \mathbb{A} -module.

Proof. First part comes directly from the fact that they have the same dimension:

$$\begin{aligned}
\dim \text{Rad}(\Delta(\lambda)) &= \dim \Delta(\lambda) - \dim L(\lambda), \\
&= \sum_{i \in I} \dim V(\lambda, i) - \text{rank} \left(\bigoplus_{i \in I_\lambda} M(\lambda, i) \right) \\
&= \sum_{i \in I_\lambda} (\dim V(\lambda, i) - \text{rank } M(\lambda, i)), \\
&= \sum_{i \in I_\lambda} \dim \text{Rad}(V(\lambda, i)).
\end{aligned}$$

Note that $V(\lambda, i) = \{0\}$ if i is not in I_λ .

Next part is coming from the fact that the basis \mathbf{B} of the module $\Delta(\lambda)$ equals $\Pi_{i \in I} \mathbf{B}_i$ and $\mathbf{B}_i = \{c_s^\lambda \mid s \in T(\lambda, i)\}$ is a basis the module $V(\lambda, i)$, also $\langle c_s^\lambda, c_t^\lambda \rangle = 0$ whenever $s \in T(\lambda, i)$ and $t \in T(\lambda, j)$ such that $i \neq j$. Let $x \in \text{Rad}(V(\lambda, i))$ for some $i \in I_\lambda$, so $\langle c_s^\lambda, x \rangle = 0$ for all $s \in T(\lambda, i)$. Moreover, it is clear that $\langle c_t^\lambda, x \rangle = 0$ for all $t \in T(\lambda, j)$ where $i \neq j$, then $x \in \text{Rad}(\Delta(\lambda))$. Thus

$$\sum_{i \in I_\lambda} \text{Rad}(V(\lambda, i)) \subseteq \text{Rad}(\Delta(\lambda)),$$

but both of them have the same dimension thus they are identical. \square

Corollary 4.4. *Let $\lambda \in \Lambda^0$, then*

$$L(\lambda) \cong \sum_{i \in I_\lambda} \bar{V}(\lambda, i),$$

as an \mathbb{A} -module.

Proof. As $V(\lambda, i) \cap V(\lambda, j) = \{0\}$ whenever $i \neq j$, so

$$L(\lambda) = \sum_{i \in I_\lambda} \frac{V(\lambda, i)}{\text{Rad}(V(\lambda, i))} \cong \sum_{i \in I_\lambda} \bar{V}(\lambda, i). \quad \square$$

5 The block decomposition of \mathbb{A}

The aim of this section is to describe the blocks of the algebra \mathbb{A} over a field \mathbb{F} by studying the homomorphisms between cell modules of \mathbb{A} .

We say $\lambda \in \Lambda$ and $\mu \in \Lambda^0$ are cell-linked if $d_{\lambda\mu} \neq 0$. A cell-block of \mathbb{A} is an equivalence class of the equivalence relation on Λ generated by this cell-linkage. From Theorem 2.2, each block of \mathbb{A} is an intersection of a cell-block with Λ^0 , see [1]. Thus, if there a non-zero homomorphism between $\Delta(\lambda)$ and $\Delta(\mu)$ where $\lambda, \mu \in \Lambda^0$, then they belong to the same block.

Let $\theta : \Delta(\lambda) \rightarrow \Delta(\mu)$ be a homomorphism defined by $c_s^\lambda \mapsto \sum_{u \in T(\mu)} \alpha_u c_u^\mu$. Now if $s \in T(\lambda, i)$ for some $i \in I$, then $u \in T(\mu, i)$ since $\theta(c_s^\lambda) = \theta(e_i c_s^\lambda) = \sum_{u \in T(\mu)} \alpha_u e_i c_u^\mu$, so

$$\theta(c_s^\lambda) = \sum_{u \in T(\mu, i)} \alpha_u c_u^\mu.$$

Hence the map θ can be restricted to define a homomorphism

$$\theta \downarrow_{e_i \mathbb{A} e_i}: V(\lambda, i) \rightarrow V(\mu, i)$$

Now if $\theta \neq 0$, then there is c_s^λ such that $\theta(c_s^\lambda) \neq 0$. Assume that $s \in T(\lambda, i)$ for some i , then $\theta \downarrow_{e_i \mathbb{A} e_i} \neq 0$, which means that both the sets $T(\lambda, i), T(\mu, i)$ don't equal the empty set.

Let $\lambda, \mu \in \Lambda_i$ for some i , and $\tau: V(\lambda, i) \rightarrow V(\mu, i)$ be a homomorphism $e_i \mathbb{A} e_i$ -modules. By extending the map τ , we obtain a homomorphism $\tau \uparrow^{\mathbb{A}}: \Delta(\lambda) \rightarrow \Delta(\mu)$. Thus

$$\text{Hom}_{\mathbb{A}}(\Delta(\lambda), \Delta(\mu)) = \{0\}$$

if and only if

$$\text{Hom}_{e_i \mathbb{A} e_i}(V(\lambda, i), V(\mu, i)) = \{0\}$$

for each $i \in I$. From this fact, we obtain the next theorem.

Theorem 5.1. *Let $\Lambda = \Lambda^0$. Two weights λ and μ in Λ are in the same block of \mathbb{A} if and only if there exist ν_0, \dots, ν_r in Λ such that all the following hold:*

1. λ and ν_0 are in the same cell-block of $e_i \mathbb{A} e_i$ for some $i \in I$.
2. For each $j = 0, \dots, r-1$, ν_j and ν_{j+1} are in the same cell-block of $e_i \mathbb{A} e_i$ for some $i \in I$.
3. μ and ν_r are in the same cell-block of $e_i \mathbb{A} e_i$ for some $i \in I$.

6 Examples

In this section, we use some simple example to illustrate the facts that have been showed in the previous sections.

Let $\mathbb{A} = M_{n \times n}(\mathbb{F})$ be an $n \times n$ matrix algebra over \mathbb{F} . This algebra is cellular with indexing set $\Lambda = \{n\}$ and $I = T(n) = \{1, \dots, n\}$. For each $i, j \in T(n)$, we take $c_{ij}^n = E_{ij}$ where E_{ij} is the matrix with 1 at the (i, j) -entry and 0 elsewhere. As we have $1_{\mathbb{A}} = \sum_{i \in I} E_{ii}$ and the elements E_{ii} satisfy all the assumptions in 3.1, thus we can apply our results from the previous sections. Note that $E_{ii} \mathbb{A} E_{ii}$ is isomorphic to \mathbb{F} for each i , so \mathbb{A} is semi-simple see Theorem 4.1.

For the second example, let \mathbb{A} be the algebra which is given by the quiver

$$\begin{array}{ccc} & a_{12} & \\ 1 & \xleftrightarrow{\quad} & 2 \\ & a_{21} & \end{array}$$

with the relation $a_{12}a_{21}a_{12} = a_{21}a_{12}a_{21} = 0$. The algebra is spanned by the elements

$$e_1, e_2, a_{12}, a_{21}, a_{12}a_{21}, a_{21}a_{12},$$

where e_i is the path of length zero on the vertex i . As left module \mathbb{A} is isomorphic to

$$\mathbb{F}\langle e_1, a_{21}, a_{12}a_{21} \rangle \oplus \mathbb{F}\langle e_2, a_{12}, a_{21}a_{12} \rangle.$$

The algebra \mathbb{A} is a cellular algebra with anti-automorphism defined by $a_{ij}^* = a_{ji}$ and $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ where $\lambda_0 > \lambda_1 > \lambda_2$ and

$$T(\lambda_0) = \{1\}, \quad T(\lambda_1) = \{1, 2\}, \quad T(\lambda_2) = \{2\}.$$

We define

$$c_{11}^{\lambda_0} = a_{12}a_{21}, \quad \begin{pmatrix} c_{11}^{\lambda_1} & c_{12}^{\lambda_1} \\ c_{21}^{\lambda_1} & c_{22}^{\lambda_1} \end{pmatrix} = \begin{pmatrix} e_1 & a_{12} \\ a_{21} & a_{21}a_{12} \end{pmatrix}, \quad c_{22}^{\lambda_2} = e_2.$$

The set $\mathcal{C} = \{c_{st}^\lambda \mid s, t \in T(\lambda) \text{ for some } \lambda \in \Lambda\}$ is a cellular basis of \mathbb{A} . Note that λ_0 is not in Λ^0 although $\Delta(\lambda_0)$ is simple.

The identity $1_{\mathbb{A}}$ equals $e_1 + e_2$ and this decomposition satisfies all the conditions in Assumption 3.1. Also we have

$$\begin{aligned} e_1 \mathbb{A} e_1 &= \mathbb{F}\langle e_1, a_{12}a_{21} \rangle & \Lambda_1 &= \{\lambda_0, \lambda_1\}, \\ e_2 \mathbb{A} e_2 &= \mathbb{F}\langle e_2, a_{21}a_{12} \rangle & \Lambda_2 &= \{\lambda_1, \lambda_2\}. \end{aligned}$$

Note that $J = \mathbb{F}\langle a_{12}a_{21} \rangle$ is a nilpotent ideal of $e_1 \mathbb{A} e_1$ and $J' = \mathbb{F}\langle a_{21}a_{12} \rangle$ is a nilpotent ideal of $e_2 \mathbb{A} e_2$, so \mathbb{A} is not semi-simple, from Theorem 4.1. Let $\mathcal{B} = \{e_1, a_{21}\}$ be a basis of the module $\Delta(\lambda_1)$, so $V(\lambda_1, 1) = \mathbb{F}\langle e_1 \rangle$ and $V(\lambda_1, 2) = \mathbb{F}\langle a_{21} \rangle$. Also we have $V(\lambda_2, 1) = \{0\}$ and $V(\lambda_2, 2) = \mathbb{F}\langle e_2 \rangle$. It is easy to show that $V(\lambda_2, 2), V(\lambda_1, 2)$ are isomorphic as $e_2 \mathbb{A} e_2$, so they are cell-linked. From Theorem 5.1, the modules $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ are in the same block of \mathbb{A} . Moreover,

$$\begin{aligned} \text{Rad}(\Delta(\lambda_1)) &= \text{Rad}(\mathbb{F}\langle e_1, a_{21} \rangle) \cong \text{Rad}(V(\lambda_1, 1) + V(\lambda_1, 2)) \\ &= V(\lambda_1, 2) \cong V(\lambda_2, 2) = \Delta(\lambda_2). \end{aligned}$$

6.1 The multi-colour partition algebra

For $n \in \mathbb{N}$, the symbol \mathcal{P}_n denotes the set of all partitions of the set $\underline{n} \cup \underline{n}'$, where $\underline{n} = \{1, \dots, n\}$ and $\underline{n}' = \{1', \dots, n'\}$.

Each individual set partition can be represented by a graph, as it is described in [5]. Any diagrams are regarded as the same diagram if they representing the same partition.

Now the composition $\beta \circ \alpha$ in \mathcal{P}_n , where $\alpha, \beta \in \mathcal{P}_n$, is the partition obtained by placing α above β , identifying the bottom vertices of α with the top vertices of β , and ignoring any connected components that are isolated from boundaries. This product on \mathcal{P}_n is associative and well-defined up to equivalence.

A (n_1, n_2) -partition diagram for any $n_1, n_2 \in \mathbb{N}^+$ is a diagram representing a set partition of the set $\underline{n_1} \cup \underline{n_2}'$ in the obvious way.

The product on \mathcal{P}_n can be generalised to define a product of (n, m) -partition diagrams when it is defined. For example, see the following figure.

Let n, m be positive integers, $\mathfrak{C}_0, \dots, \mathfrak{C}_{m-1}$ be different colours where none of them is white, and $\delta_0, \dots, \delta_{m-1}$ be scalars corresponding to these colours.

Define the set $\Phi^{n,m}$ to be

$$\{(A_0, \dots, A_{m-1}) \mid \{A_0, \dots, A_{m-1}\} \in \mathcal{P}_n\}.$$

Let $(A_0, \dots, A_{m-1}) \in \Phi^{n,m}$ (note that some of these subsets can be an empty set). Define $\mathcal{P}_{A_0, \dots, A_{m-1}}$ to be the set $\prod_{i=0}^{m-1} \mathcal{P}_{A_i}$, where \mathcal{P}_{A_i} is the set of all partitions of A_i , and

$$\mathcal{P}_{n,m} := \bigcup_{(A_0, \dots, A_{m-1}) \in \Phi^{n,m}} \mathcal{P}_{A_0, \dots, A_{m-1}}.$$

The element $d = (d_0, \dots, d_{m-1}) \in \prod_{i=0}^{m-1} \mathcal{P}_{A_i}$ can be represented by the same diagram of the partition $\cup_{i=0}^{m-1} d_i \in \mathcal{P}_n$ after colouring it as follows: we use the colour \mathfrak{C}_i to draw all the edges and the nodes in the partition d_i .

A diagram represents an element in $\mathcal{P}_{n,m}$ is not unique. We say two diagrams are equivalent if they represent the same tuple of partitions. The term multi-colour partition diagram will be used to mean an equivalence class of a given diagram. For example, the following diagrams



are equivalent.

We define the following sets for each element $d \in \prod \mathcal{P}_{A_i}$:

$$\begin{aligned} \text{top}(d_i) &= A_i \cap \underline{n}, & \text{bot}(d_i) &= A_i \cap \underline{n}', \\ \text{top}(d) &= (\text{top}(d_0), \dots, \text{top}(d_{m-1})), & \text{bot}(d) &= (\text{bot}(d_0), \dots, \text{bot}(d_{m-1})). \end{aligned}$$

Let $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ be \mathbb{F} -vector space with the basis $\mathcal{P}_{n,m}$, as it is defined in [3], and with the composition:

$$(\alpha_i)(\beta_i) = \begin{cases} \prod_{i=0}^{m-1} \delta_i^{c_i}(\beta_j \circ \alpha_j) & \text{if } \text{bot}(\alpha) = \text{top}(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

where $\delta_i \in \mathbb{F}$, $\alpha, \beta \in \mathcal{P}_{n,m}$, c_i is the number of removed connected components from the middle row when computing the product $\beta_i \circ \alpha_i$ for each $i = 0, \dots, m-1$ and \circ is the normal composition of partition diagrams.

The vector space $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ is an associative algebra, called the *multi-colour partition algebra*, with identity:

$$1_{\mathbb{P}_{n,m}} = \sum_{(A_0, \dots, A_{m-1}) \in \Xi^{n,m}} 1_{(A_0, \dots, A_{m-1})} := \sum_{(A_0, \dots, A_{m-1}) \in \Xi^{n,m}} (1_{A_0}, \dots, 1_{A_{m-1}}),$$

where $\Xi^{n,m} := \{(A_0, \dots, A_{m-1}) \mid \cup_{i=0}^{m-1} A_i = \underline{n}, A_i \cap A_j = \emptyset \forall i \neq j\}$, 1_{A_i} is the partition of the set $A_i \cup A'_i$ where any node j is only connected with the node j' for all $j \in A_i$ and $A'_i = \{j' \mid j \in A_i\}$, for all $0 \leq i \leq m-1$. This means the identity is the summation of all the different multi-colour partitions that their diagrams connect i only to i' with any colour for each $1 \leq i \leq n$.

The diagrams of shape $id \in \mathfrak{S}_n$ are orthogonal idempotents, since

$$1_{(A_0, \dots, A_{m-1})} 1_{(B_0, \dots, B_{m-1})} = \begin{cases} 0 & \text{if } (A_i) \neq (B_i), \\ 1_{(A_0, \dots, A_{m-1})} & \text{if } (A_i) = (B_i), \end{cases}$$

for all $(A_i), (B_i) \in \Xi^{n,m}$. Thus we have a decomposition of the identity as a sum of orthogonal idempotents since $1_{\mathbb{P}_{n,m}} = \sum_{(A_i) \in \Xi^{n,m}} 1_{(A_0, \dots, A_{m-1})}$. As it have been showed in

[3], the algebra $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ is cellular and the last decomposition satisfies all the conditions in Assumption 3.1.

Let $(A_i) \in \Xi^{n,m}$, then

$$1_{(A_i)} \mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1}) 1_{(A_i)} \cong \mathbb{P}_{|A_0|}(\delta_0) \otimes_{\mathbb{F}} \dots \otimes_{\mathbb{F}} \mathbb{P}_{|A_{m-1}|}(\delta_{m-1}), \quad (6)$$

where $\mathbb{P}_{|A_i|}(\delta_i)$ is the normal partition algebra and $|A_i|$ is the cardinality of A_i , for the proof see Chapter 2 in [3].

Theorem 6.1. *The algebra $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ is semisimple over \mathbb{C} for each integers $n \geq 0$ and $m \geq 1$ if and only if none of the parameters δ_i is a natural number less than $2n$.*

Proof. As the algebra $\mathbb{P}_n(\delta)$ is semi-simple over \mathbb{C} whenever δ is not an integer in the range $[0, 2n - 1]$, see Corollary 10.3 in [7], we obtain this theorem. \square

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